Multidimensional Social Welfare Dominance with 4th Order Derivatives of Utility

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1. Dominance

- Poverty, Inequality, Social Welfare
- Robust ‘dominance’ judgments: accepted by people with different norms
- One-dimensional settings: H-L-P (1929), Karamata (1932), Kolm (1969) and Atkinson (1970)
- Normative hypotheses: e.g., variations in aversion to inequality

Multi-Dimensional Setting

- Less obvious how to obtain powerful rules
- Atkinson and Bourguignon Restud82, 87; Koshevoy 95, JASA98; Moyes 99
- Bazen and Moyes 03, Gravel and Moyes 12, Muller and Trannoy JET11, 12
- etc
- Signs of 4^{th} order derivatives generally not used because believed to be hard to interpret
- How to gain discriminatory power?

The Contribution

- A new method to incorporate normative restrictions in welfare analysis:

  ‘*Welfare Shock Sharing*’
- Providing normative interpretations to sign conditions for 4^{th} degree derivatives of utility
- Characterization of a new asymmetric condition: $U_{1112} < 0$
- New Necessary and Sufficient condition for SD results for several classes of utilities
- Poverty Ordering characterizations

Signs of derivatives of utility

- Two attributes
- Signs of derivatives of utility as normative conditions
- $\Delta W = W_F - W_{F*} = \int\int U(x, y)\, d\, \Delta F(x, y)$
- Continuous distributions
- U defined and 'sufficiently' differentiable over

x in]0, a_1] and y in]0, a_2]; or any intervals

- Benchmark: $U_1, U_2 \geq 0$; $U_{12}, U_{11}, U_{22} \leq 0$
- $U_{111}, U_{112}, U_{122}, U_{222} \geq 0$
- Not always necessary to assume all of the above
- $U_{1111}, U_{1112}, U_{1122}, U_{1222}, U_{2222} \leq 0$

2. Welfare Shock Sharing

- Extending welfare notions by defining ‘Social Shocks’ and stating solidarity
- Take two individuals with same bivariate non-random endowments. Which welfare effect of some welfare shocks on this small society?
- Welfare shocks may be: losses of some attributes, risks affecting some attributes,…
- Applications to SWFs additive in individual utility functions of possibly random variables

Let be any endowments $(x,y) \in R_+^2$. Let c and d > 0. Let ε be a centered real random variable and δ be a centered real random variable independent of ε

- (i) A social planner is said to be *Welfare Correlation Averse* if x-c > 0 and y-d > 0 implies that the social planner prefers the state

{(x-c,y);(x,y-d)} to the state {(x,y);(x-c,y-d)}

That is: *`Sharing fixed losses affecting different attributes improves social welfare*'

- (ii) A social planner is said to be *Welfare Prudent* in x if $x+\varepsilon > 0$ and $x-c > 0$ implies that the planner prefers the state

$$\{(x-c,y);(x+\varepsilon,y)\} \text{ to } \{(x-c+\varepsilon,y);(x,y)\}$$

`*Sharing a fixed loss and a centred risk affecting the same first attribute improves social welfare* '

- (iii) A social planner is said to be *Welfare Cross-Prudent* in x if $y+\delta > 0$ and $x-c > 0$ implies that the planner prefers the state

$$\{(x,y+\delta);(x-c,y)\} \text{ to } \{(x,y);(x-c,y+\delta)\}$$

`*Sharing a fixed loss and a centred risk affecting different attributes improves social welfare* '

- (iv) A social planner is said to be *Welfare Temperate* in x if $x+\varepsilon > 0$, $x+\delta > 0$ and $x+\delta+\varepsilon > 0$ implies that the planner prefers the state

$$\{(x+\delta,y);(x+\varepsilon,y)\} \text{ to } \{(x,y);(x+\delta+\varepsilon,y)\}$$

`*Sharing centred risks affecting the same first attribute improves social welfare*'

- (v) A social planner is said to be *Welfare Cross-Temperate* if $x+\varepsilon > 0$ and $y+\delta > 0$ implies that the planner prefers the state
  - $\{(x+\varepsilon,y);(x,y+\delta)\}$ to $\{(x,y);(x+\varepsilon,y+\delta)\}$

`*Sharing centred risks affecting different attributes improves social welfare*'

- (vi) A social planner is said to be *Welfare-Premium Correlation Averse* in x if $x+\varepsilon > 0$,

$x-c+\varepsilon > 0$ and $y-d > 0$ implies that the planner prefers the state

$$\{(x-c,y);(x,y-d);\ (x+\varepsilon,y);(x+\varepsilon-c,y-d)\}$$

$$\text{to } \{(x,y);(x-c,y-d);\ (x+\varepsilon-c,y);(x+\varepsilon,y-d)\}$$

`*Sharing fixed losses affecting different attributes improves social welfare, while less so under background risk in the first attribute*'

Equivalences under Expected Utility

- (a) *Inequality Aversion* is equivalent to $U_{11} \leq 0$ (Eq^{t} to preference for sharing fixed losses in x)
- (b) *Welfare Correlation Aversion* is equivalent to $U_{12} \leq 0$
- (c) *Welfare Prudence* in x is equivalent to $U_{111} \geq 0$
- (d) *Welfare Temperance* in x is equivalent to $U_{1111} \leq 0$
- (e) *Welfare Cross-Prudence* in x is equivalent to $U_{122} \geq 0$
- (f) *Welfare Cross-Temperance* is equivalent to $U_{1122} \leq 0$
- (g) *Welfare Premium Correlation Aversion* in x is equivalent to $U_{1112} \leq 0$

Proof for $U_{1112} \leq 0$

- Let c be a fixed loss and ε be a centred risk
- Jensen's gap for a function w:

Let $v(x,y) = w(x,y;c) - Ew(x+\varepsilon,y;c)$,

where $w(x,y;c) = U(x,y) - U(x-c,y)$ = Utility loss due to a fall in the first attribute.

- Then, $v_2(x,y) = w_2(x,y;c) - Ew_2(x+\varepsilon,y;c) \leq 0$

iff $w_{112} \leq 0$, that is: $\mathbf{U_{1112} \leq 0}$

Because same sign for derivatives and finite variations

- $v_2(x,y) = w_2(x,y;c) - Ew_2(x+\varepsilon,y;c) \leq 0$, all c

iff $w(x,y;c) - Ew(x+\varepsilon,y;c) - w(x,y-d;c)$

$+ Ew(x+\varepsilon,y-d;c) \leq 0$, for all c and d

- Then, $U(x,y) - U(x-c,y) - EU(x+\varepsilon,y)$

$+ EU(x-c+\varepsilon,y) - U(x,y-d) + U(x-c,y-d)$

$+ EU(x+\varepsilon,y-d) - EU(x-c+\varepsilon,y-d) \leq 0$

Therefore, for a 4-person society:

- **$U(x-c,y) + U(x,y-d) + EU(x+\varepsilon,y) + EU(x-c+\varepsilon,y-d)$ $\geq U(x,y) + U(x-c,y-d) + EU(x-c+\varepsilon,y) + EU(x+\varepsilon,y-d)$**
- Interpretation by decomposing in two groups

- **$\{(x-c,y); (x,y-d); (x+\varepsilon,y); (x+\varepsilon-c,y-d)\}$**

preferred to

$\{(x,y); (x-c,y-d); (x+\varepsilon-c,y); (x+\varepsilon,y-d)\}$

- Utility Premium $p^x(x,y,\varepsilon) = U(x,y) - EU(x+\varepsilon,y)$

- Premium for being an individual under risk rather than another without risk, under veil of ignorance

$$p^x(x-c,y,\varepsilon) + p^x(x,y-d,\varepsilon)$$

is preferred to

$$p^x(x,y,\varepsilon) + p^x(x-c,y-d,\varepsilon)$$

- ‘*Welfare-Premium Correlation Aversion*’

3. Stochastic Dominance

- ‘*(s_1,s_2)-icv: (s_1,s_2)-increasing concave*’:

$$(-1)^{k_1+k_2+1}\,[\partial^{k_1+k_2}/\partial^{k_1}x\,\partial^{k_2}y]\,g \geq 0$$

for $k_i = 0,\ldots, s_i$; $i = 1, 2$; s_i non-negative integers and $1 \leq k_1+k_2$

- ‘*s-idircv*: *s-increasing directionally concave*

$$(-1)^{k_1+k_2+1}\,[\partial^{k_1+k_2}/\partial^{k_1}x\,\partial^{k_2}y]\,g \geq 0$$

for k_1 and k_2 non-negative integers and

$1 \leq k_1+k_2 \leq s$, s is a non-negative integer ≥ 2

- Let s be an integer greater of equal to n
- $R_s = \{(r_1, r_2) \in N^2 | 1 \leq r_1 + r_2 = s\}$

- Let U_S be the set of generators of a set of utility functions S. Then,

$$U_{s\text{-idircv}} = \bigcap_{\{(r_1,r_2) \in Rs\}} U_{(r_1,r_2)\text{-icv}}$$

- $H_x(x) = \int_0^x F_x(s)ds$
- $L_x(x) = \int_0^x\int_0^t F_x(s)dsdt$
- $M_x(x) = \int_0^x\int_0^u\int_0^t F_x(s)dsdtdu$

- $H(x,y) = \int_0^x\int_0^y F(s,t)dsdt$
- $H_x(x; y) = \int_0^x F(s,y)ds$
- $L_x(x; y) = \int_0^x\int_0^s F(u,y)duds$
- $M_x(x; y) = \int_0^x\int_0^s\int_0^u F(t,y)dtduds$

- Idem by substituting the roles of x and y

Stochastic Dominance Results

- For any distributions : $\Delta F = F - F^*$
- All usual signs for first and second derivatives of utility
- (A&B82):$1^{st}+2^{nd}+U_{112},U_{122} \geq 0$, $U_{1122} \leq 0$

  F SD F^* is equivalent to:

(1) For all x , $\Delta H_x(x) \leq 0$

(2) For all y , $\Delta H_y(y) \leq 0$

(3) For all x, y , $\Delta H(x, y) \leq 0$

- Now a full proof of NSC

(3,1)-icv: $U_1, U_2 \geq 0$; $U_{11}, U_{12} \leq 0$; $U_{112}, U_{111} \geq 0$; $U_{1112} \leq 0$

- (a) $\Delta L_x(x; y) \leq 0$, for all x, y
- (b) $\Delta H_x(a_1; y) \leq 0$, for all y
- (c) $\Delta F_y(y) \leq 0$, for all y
- Idem for (1,3)-icv

4-icv:

$U_1 \geq 0$; $U_{11} \leq 0$; $U_{111} \geq 0$; $U_{1111} \leq 0$

- One-dimensional: results already known (4^{th} degree SD)
- NOW there is a good reason to assume $U_{1111} \leq 0$: 'Sharing risks on x is good for social welfare'

- (a) $\Delta M_x(x) \leq 0$, for all x
- (b) $\Delta L_x(a_1) \leq 0$
- (c) $\Delta H_x(a_1) \leq 0$

- Idem with y

4-idircv: $U_1, U_2 \geq 0$; $U_{11}, U_{12}, U_{22} \leq 0$; $U_{111}, U_{112}, U_{122}, U_{222} \geq 0$; $U_{1111}, U_{1222}, U_{1122}, U_{1112}, U_{2222} \leq 0$

- Has a class of generators that is the intersection of the classes of generators of

  the (s_1, s_2)-icv functions sets

  with (s_1, s_2) in $\{(2,2),(3,1),(1,3),(4,0),(0,4)\}$

- So far, the generators of this class were not known

Change in variable in the complex plan

- $z = x + i\,y = \rho\, e^{i\theta}$
- Modulus $\rho = |z| = \text{sqrt}\,(x^2 + y^2)$
- $\theta = \text{Arg}\, z$ in $[0, \pi/2]$ since $x, y > 0$

- *Theorem:*

4-idircv in (x,y) is equivalent to 4-icv in ρ

4-idircv Stochastic Dominance

NSC with $a_1 = a_2 = +\infty$:

- *(a)* $\Delta M_\rho(\rho) \leq 0$, *for all* ρ
- *(b)* $\Delta L_\rho(+\infty) \leq 0$
- *(c)* $\Delta H_\rho(+\infty) \leq 0$
- *An appropriate bound* a_ρ *for (b) and (c) in the cases with bounded domains*
- *Examples of various domains for (x,y)*

Generators of 4-idircv

- The generators of the 4-idircv class are the functions of x and y defined by:

$$\text{Max}\{c - \text{sqrt}(x^2+y^2),0\}^{k-1},$$

- for all $c \in [0, a_\rho]$, if $k= 4$ and $c = a_\rho$ if $k=1,2,3$

4. Poverty Orderings

- $P^{k_1,k_2} = \int_{[0,z_2]}\int_{[0,z_1]} (z_1\text{-}x)^{k_1\text{-}1}(z_2\text{-}y)^{k_2\text{-}1}\, dF(x,y)$

- 4-icv (in x) dominance ordering is equivalent to the poverty ordering $P^4(z_x) = P^{4,0}(z_x$, y_max) in x

+ SSD and TSD conditions at bounds

- 4-idircv dominance ordering is equivalent to the poverty ordering $P^4(z_\rho)$ in ρ

+ SSD and TSD conditions at bounds

- (3,1)-icv dominance ordering is equivalent to the poverty ordering $P^{k_1,1}$

for all $z_x \in [0, x\_max]$ if $k_1=3$ and $z_x = x\_max$ if $k_1=1,2$; and $z_y = y\_max$ with $k_2=1$

- (2,2)-icv dominance ordering is equivalent to the poverty ordering P^{k_1,k_2}

for all $z_x \in [0, x\_max]$ if $k_1=2$ and $z_x = x\_max$ if $k_1=1$; and idem for k_2 and y

5. Conclusion

- A new normative approach:
  Welfare Shock Sharing

- Normative interpretations of the signs of 4^{th} degree derivatives of utilities

- A new characterization for $U_{1112} < 0$

- *Necessary and Sufficient SD results* for several classes of functions
- Equivalence with multivariate poverty orderings

- To finish: Empirical application

- To come: More dimensions and higher degree
- To come: Generalised polar stochastic dominance
- More on risk analysis