Multidimensional Social Welfare Dominance with 4th Order Derivatives of Utility

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1. Dominance

- Poverty, Inequality, Social Welfare

- Robust ‘dominance’ judgments: accepted by people with different norms


- Normative hypotheses: e.g., variations in aversion to inequality
Multi-Dimensional Setting

• Less obvious how to obtain powerful rules
• Atkinson and Bourguignon Restud82, 87; Koshevoy 95, JASA98; Moyes 99
• Bazen and Moyes 03, Gravel and Moyes 12, Muller and Trannoy JET11, 12
• etc
• Signs of 4th order derivatives generally not used because believed to be hard to interpret
• How to gain discriminatory power?
The Contribution

• A new method to incorporate normative restrictions in welfare analysis: ‘Welfare Shock Sharing’
• Providing normative interpretations to sign conditions for 4\textsuperscript{th} degree derivatives of utility
• Characterization of a new asymmetric condition: $U_{1112} < 0$
• New Necessary and Sufficient condition for SD results for several classes of utilities
• Poverty Ordering characterizations
Signs of derivatives of utility

- Two attributes
- Signs of derivatives of utility as normative conditions
- \( \Delta W = W_F - W_{F^*} = \int \int U(x, y) \, d \Delta F(x, y) \)
- Continuous distributions
- \( U \) defined and ‘sufficiently’ differentiable over \( x \) in \( ]0, a_1[ \) and \( y \) in \( ]0, a_2[ \); or any intervals
- Benchmark: \( U_1, U_2 \geq 0; U_{12}, U_{11}, U_{22} \leq 0 \)
- \( U_{111}, U_{112}, U_{122}, U_{222} \geq 0 \)
- Not always necessary to assume all of the above
- \( U_{1111}, U_{1112}, U_{1122}, U_{1222}, U_{2222} \leq 0 \)
2. Welfare Shock Sharing

- Extending welfare notions by defining ‘Social Shocks’ and stating solidarity
- Take two individuals with same bivariate non-random endowments. Which welfare effect of some welfare shocks on this small society?
- Welfare shocks may be: losses of some attributes, risks affecting some attributes,…
- Applications to SWFs additive in individual utility functions of possibly random variables
Let be any endowments \((x, y) \in \mathbb{R}_+^2\). Let \(c\) and \(d > 0\). Let \(\varepsilon\) be a centered real random variable and \(\delta\) be a centered real random variable independent of \(\varepsilon\).

• (i) A social planner is said to be \textit{Welfare Correlation Averse} if \(x-c > 0\) and \(y-d > 0\) implies that the social planner prefers the state \{
\(x-c, y\); \((x, y-d)\)\} to the state \{
\((x, y); (x-c, y-d)\)\}.

That is: `Sharing fixed losses affecting different attributes improves social welfare'
• (ii) A social planner is said to be Welfare Prudent in $x$ if $x+\varepsilon > 0$ and $x-c > 0$ implies that the planner prefers the state

$$\{(x-c,y);(x+\varepsilon,y)\} \text{ to } \{(x-c+\varepsilon,y);(x,y)\}$$

`Sharing a fixed loss and a centred risk affecting the same first attribute improves social welfare`'

• (iii) A social planner is said to be Welfare Cross-Prudent in $x$ if $y+\delta > 0$ and $x-c > 0$ implies that the planner prefers the state

$$\{(x,y+\delta);(x-c,y)\} \text{ to } \{(x,y);(x-c,y+\delta)\}$$

`Sharing a fixed loss and a centred risk affecting different attributes improves social welfare`'
• (iv) A social planner is said to be *Welfare Temperate* in $x$ if $x+\varepsilon > 0$, $x+\delta > 0$ and $x+\delta+\varepsilon > 0$ implies that the planner prefers the state

$$\{(x+\delta, y); (x+\varepsilon, y)\}$$
to $$\{(x, y); (x+\delta+\varepsilon, y)\}$$
`Sharing centred risks affecting the same first attribute improves social welfare`

• (v) A social planner is said to be *Welfare Cross-Temperate* if $x+\varepsilon > 0$ and $y+\delta > 0$ implies that the planner prefers the state

$$\{(x+\varepsilon, y); (x, y+\delta)\}$$
to $$\{(x, y); (x+\varepsilon, y+\delta)\}$$
`Sharing centred risks affecting different attributes improves social welfare`
(vi) A social planner is said to be Welfare-Premium Correlation Averse in $x$ if $x+\varepsilon > 0$, $x-c+\varepsilon > 0$ and $y-d > 0$ implies that the planner prefers the state

\[
\{(x-c,y);(x,y-d); (x+\varepsilon,y);(x+\varepsilon-c,y-d)\}
\]

to \{(x,y);(x-c,y-d); (x+\varepsilon-c,y);(x+\varepsilon,y-d)\}

`Sharing fixed losses affecting different attributes improves social welfare, while less so under background risk in the first attribute`
Equivalences under Expected Utility

• (a) *Inequality Aversion* is equivalent to $U_{11} \leq 0$ (Eq to preference for sharing fixed losses in $x$)
• (b) *Welfare Correlation Aversion* is equivalent to $U_{12} \leq 0$
• (c) *Welfare Prudence* in $x$ is equivalent to $U_{111} \geq 0$
• (d) *Welfare Temperance* in $x$ is equivalent to $U_{1111} \leq 0$
• (e) *Welfare Cross-Prudence* in $x$ is equivalent to $U_{122} \geq 0$
• (f) *Welfare Cross-Temperance* is equivalent to $U_{1122} \leq 0$
• (g) *Welfare Premium Correlation Aversion* in $x$ is equivalent to $U_{1112} \leq 0$
Proof for $U_{1112} \leq 0$

- Let $c$ be a fixed loss and $\varepsilon$ be a centred risk.
- Jensen’s gap for a function $w$:
  Let $v(x,y) = w(x,y;c) - Ew(x+\varepsilon,y;c)$,
  where $w(x,y;c) = U(x,y) - U(x-c,y) = $ Utility loss due to a fall in the first attribute.
- Then, $v_2(x,y) = w_2(x,y;c) - Ew_2(x+\varepsilon,y;c) \leq 0$
  iff $w_{112} \leq 0$, that is: $U_{1112} \leq 0$

Because same sign for derivatives and finite variations.
• \( v_2(x, y) = w_2(x, y; c) - E w_2(x+\varepsilon, y; c) \leq 0, \text{ all } c \) iff \( w(x, y; c) - E w(x+\varepsilon, y; c) - w(x, y-d; c) + E w(x+\varepsilon, y-d; c) \leq 0, \text{ for all } c \) and \( d \)

• Then, \( U(x, y) - U(x-c, y) - E U(x+\varepsilon, y) + E U(x-c+\varepsilon, y) - U(x, y-d) + U(x-c, y-d) + E U(x+\varepsilon, y-d) - E U(x-c+\varepsilon, y-d) \leq 0 \)

Therefore, for a 4-person society:

• \( U(x-c, y) + U(x, y-d) + E U(x+\varepsilon, y) + E U(x-c+\varepsilon, y-d) \geq U(x, y) + U(x-c, y-d) + E U(x-c+\varepsilon, y) + E U(x+\varepsilon, y-d) \)

• Interpretation by decomposing in two groups
• \{ (x-c, y); (x, y-d); (x+\varepsilon, y); (x+\varepsilon-c, y-d) \}
preferred to
\{ (x, y); (x-c, y-d); (x+\varepsilon-c, y); (x+\varepsilon, y-d) \}

• Utility Premium $p^x(x, y, \varepsilon) = U(x, y) - EU(x+\varepsilon, y)$

• Premium for being an individual under risk rather than another without risk, under veil of ignorance

$$p^x(x-c, y, \varepsilon) + p^x(x, y-d, \varepsilon)$$
is preferred to
$$p^x(x, y, \varepsilon) + p^x(x-c, y-d, \varepsilon)$$

• ‘Welfare-Premium Correlation Aversion’
3. Stochastic Dominance

• ‘\((s_1,s_2)\text{-icv}: (s_1,s_2)\text{-increasing concave}\)’:
  \[-1]^{k_1+k_2+1} \left[ \frac{\partial^{k_1+k_2}}{\partial^{k_1}x \partial^{k_2}y} \right] g \geq 0
  \text{ for } k_i = 0,\ldots, s_i; \ i = 1, 2; \ s_i \text{ non-negative integers and } 1 \leq k_1+k_2

• ‘\(s\text{-idircv: s-increasing directionally concave}\)’:
  \[-1]^{k_1+k_2+1} \left[ \frac{\partial^{k_1+k_2}}{\partial^{k_1}x \partial^{k_2}y} \right] g \geq 0
  \text{ for } k_1 \text{ and } k_2 \text{ non-negative integers and } 1 \leq k_1+k_2 \leq s, \ s \text{ is a non-negative integer } \geq 2
• Let $s$ be an integer greater of equal to $n$

• $R_s = \{(r_1, r_2) \in \mathbb{N}^2 \mid 1 \leq r_1 + r_2 = s\}$

• Let $U_S$ be the set of generators of a set of utility functions $S$. Then,

$$U_{s-idircv} = \bigcap \{(r_1, r_2) \in R_s\} U_{(r_1, r_2)-icv}$$
• \( H_x(x) = \int_0^x F_x(s)ds \)
• \( L_x(x) = \int_0^x \int_0^t F_x(s)dsdt \)
• \( M_x(x) = \int_0^x \int_0^u \int_0^t F_x(s)dsdtdu \)

• \( H(x,y) = \int_0^x \int_0^y F(s,t)dsdt \)
• \( H_x(x; y) = \int_0^x F(s,y)ds \)
• \( L_x(x; y) = \int_0^x \int_0^s F(u,y)duds \)
• \( M_x(x; y) = \int_0^x \int_0^s \int_0^u F(t,y)dtduds \)

• Idem by substituting the roles of \( x \) and \( y \)
Stochastic Dominance Results

• For any distributions : $\Delta F = F - F^*$
• All usual signs for first and second derivatives of utility
• (A&B82): $1^{st} + 2^{nd} + U_{112}, U_{122} \geq 0, U_{1122} \leq 0$
F SD F* is equivalent to:
(1) For all $x$, $\Delta H_x(x) \leq 0$
(2) For all $y$, $\Delta H_y(y) \leq 0$
(3) For all $x, y$, $\Delta H(x, y) \leq 0$
• Now a full proof of NSC
(3,1)-icv:
\[ U_{11}, U_{12} \geq 0; U_{111}, U_{12} \leq 0; \]
\[ U_{112}, U_{111} \geq 0; U_{1112} \leq 0 \]

• (a) \( \Delta L_x(x; y) \leq 0 \), for all \( x, y \)

• (b) \( \Delta H_x(a_1; y) \leq 0 \), for all \( y \)

• (c) \( \Delta F_y(y) \leq 0 \), for all \( y \)

• Idem for (1,3)-icv
4-icv:

\[ U_1 \geq 0 ; U_{11} \leq 0; U_{111} \geq 0; U_{1111} \leq 0 \]

- One-dimensional: results already known (4th degree SD)
- NOW there is a good reason to assume \( U_{1111} \leq 0 \): ‘Sharing risks on x is good for social welfare’

- (a) \( \Delta M_x(x) \leq 0 \), for all \( x \)
- (b) \( \Delta L_x(a_1) \leq 0 \)
- (c) \( \Delta H_x(a_1) \leq 0 \)

- Idem with \( y \)
4-idircv: $U_1, U_2 \geq 0; U_{11}, U_{12}, U_{22} \leq 0; U_{111}, U_{112}, U_{122}, U_{222} \geq 0; U_{1111}, U_{1222}, U_{1122}, U_{1112}, U_{2222} \leq 0$

- Has a class of generators that is the intersection of the classes of generators of the $(s_1, s_2)$-icv functions sets with $(s_1, s_2)$ in \{(2,2),(3,1),(1,3),(4,0),(0,4)\}

- So far, the generators of this class were not known
Change in variable in the complex plan

- \( z = x + i \, y = \rho \, e^{i\theta} \)
- Modulus \( \rho = |z| = \sqrt{x^2 + y^2} \)
- \( \theta = \text{Arg } z \) in \([0, \pi/2]\) since \( x, y > 0 \)

- **Theorem:**

\( 4\text{-idircv in } (x,y) \) is equivalent to \( 4\text{-icv in } \rho \)
4-idircv Stochastic Dominance

NSC with $a_1 = a_2 = +\infty$:

- (a) $\Delta M_\rho(\rho) \leq 0$, for all $\rho$
- (b) $\Delta L_\rho(+\infty) \leq 0$
- (c) $\Delta H_\rho(+\infty) \leq 0$

- An appropriate bound $a_\rho$ for (b) and (c) in the cases with bounded domains
- Examples of various domains for $(x,y)$
Generators of 4-idircv

• The generators of the 4-idircv class are the functions of x and y defined by:

\[ \text{Max}\{c - \sqrt{x^2+y^2},0\}^{k-1}, \]

• for all \( c \in [0, a_\rho] \), if \( k = 4 \) and \( c = a_\rho \) if \( k = 1, 2, 3 \)
4. Poverty Orderings

- \( P^{k_1,k_2} = \int_{[0,z_2]} \int_{[0,z_1]} (z_1-x)^{k_1-1}(z_2-y)^{k_2-1} dF(x,y) \)

- 4-icv (in x) dominance ordering is equivalent to the poverty ordering \( P^4(z_x) = P^{4,0}(z_x, y_{\text{max}}) \) in x + SSD and TSD conditions at bounds

- 4-idircv dominance ordering is equivalent to the poverty ordering \( P^4(z_\rho) \) in \( \rho \) + SSD and TSD conditions at bounds
• (3,1)-icv dominance ordering is equivalent to the poverty ordering \( P^{k_1,1} \)

for all \( z_x \in [0, \text{x\_max}] \) if \( k_1=3 \) and \( z_x = \text{x\_max} \) if \( k_1=1,2; \) and \( z_y = \text{y\_max} \) with \( k_2=1 \)

• (2,2)-icv dominance ordering is equivalent to the poverty ordering \( P^{k_1,k_2} \)

for all \( z_x \in [0, \text{x\_max}] \) if \( k_1=2 \) and \( z_x = \text{x\_max} \) if \( k_1=1; \) and idem for \( k_2 \) and \( y \)
5. Conclusion

- A new normative approach: Welfare Shock Sharing
- Normative interpretations of the signs of 4th degree derivatives of utilities
- A new characterization for $U_{1112} < 0$
- Necessary and Sufficient SD results for several classes of functions
- Equivalence with multivariate poverty orderings
- To finish: Empirical application
- To come: More dimensions and higher degree
- To come: Generalised polar stochastic dominance
- More on risk analysis